

# RAMIFICATION ESTIMATE FOR FONTAINE-LAFFAILLE GALOIS MODULES

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**ABSTRACT.** Suppose  $K$  is unramified over  $\mathbb{Q}_p$  and  $\Gamma_K = \text{Gal}(\bar{K}/K)$ . Let  $H$  be a torsion  $\Gamma_K$ -equivariant subquotient of crystalline  $\mathbb{Q}_p[\Gamma_K]$ -module with HT weights from  $[0, p-2]$ . We give a new proof of Fontaine's conjecture about the triviality of action of some ramification subgroups  $\Gamma_K^{(v)}$  on  $H$ . The earlier author's proof from [1] contains a gap and proves this conjecture only for some subgroups of index  $p$  in  $\Gamma_K^{(v)}$ .

## INTRODUCTION

Let  $W(k)$  be the ring of Witt vectors with coefficients in a perfect field  $k$  of characteristic  $p$ . Consider the field  $K = W(k)[1/p]$ , choose its algebraic closure  $\bar{K}$  and set  $\Gamma_K = \text{Gal}(\bar{K}/K)$ . Denote by  $\mathbb{C}_p$  the completion of  $\bar{K}$  and use the notation  $O_{\mathbb{C}_p}$  for its valuation ring.

For  $a \in \mathbb{Z}_{\geq 0}$ , let  $\text{MF}_{\mathbb{Q}_p}^{\text{cr}}(a)$  be the category of crystalline  $\mathbb{Q}_p[\Gamma_K]$ -modules with Hodge-Tate weights from  $[0, a]$ . Define the full subcategory  $\text{MF}_N^{\text{cr}}(a)$  of the category of  $\Gamma_K$ -modules consisting of  $H = H_1/H_2$ , where  $H_1, H_2$  are  $\Gamma_K$ -invariant lattices in  $V \in \text{MF}_{\mathbb{Q}_p}^{\text{cr}}(a)$  and  $p^N H_1 \subset H_2 \subset H_1$ . J.-M. Fontaine conjectured in [4] that the ramification subgroups  $\Gamma_K^{(v)}$  act on  $H \in \text{MF}_N^{\text{cr}}(a)$  trivially if  $v > N - 1 + a/(p-1)$ . The author suggested in [1] a proof of this conjecture under the assumption  $0 \leq a \leq p-2$ .

It was pointed recently by Sh. Hattori to the author that the proof in [1] has a gap. More precisely, let  $R = \varprojlim (O_{\mathbb{C}_p}/p)$  be Fontaine's ring. For  $r = (o_n \bmod p)_{n \geq 0} \in R$  and  $m \in \mathbb{Z}$ , set  $r^{(m)} = \lim_{n \rightarrow \infty} o_n^{p^{n+m}} \in O_{\mathbb{C}_p}$  and consider Fontaine's map  $\gamma : W_N(R) \rightarrow O_{\mathbb{C}_p}/p^N$ , where  $(r_0, \dots, r_{N-1}) \mapsto \sum_{0 \leq i < N} p^i r_i^{(i)} \bmod p^N$ . Consider the projection  $(\bar{o}_0, \dots, \bar{o}_N, \dots) \mapsto \bar{o}_N$  from  $R$  to  $O_{\mathbb{C}_p}/p$  and denote the image of  $\text{Ker } \gamma$  in  $W_N(O_{\mathbb{C}_p}/p)$  by  $W_N^1(O_{\mathbb{C}_p}/p)$ . This is principal ideal and in order to apply Fontaine's criterion about the triviality of the action of ramification subgroups from [4], we needed an element of  $W_N(L)$ , where  $L$  is a finite extension of  $K$  with "small" ramification, which generates  $W_N^1(O_{\mathbb{C}_p}/p)$ . Our "truncation" argument in [1] does not actually work: the resulting element does not belong to  $W_N^1(O_{\mathbb{C}_p}/p)$ . In the moment

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the author is inclined to believe that such an element does not exist if  $N > 1$ . Nevertheless, our proof in [1] gives the Fontaine conjecture up to index  $p$ : the groups  $\Gamma_K^{(v)}$  just should be replaced by the groups  $\Gamma_K^{(v)} \cap \Gamma_{K(\zeta_{N+1})}$ , where  $\zeta_{N+1}$  is a primitive  $p^{N+1}$ -th root of unity.

The above difficulty appears in many other situations when we try to escape from “ $R$ -constructions” (e.g.  $W(R)$ ,  $A_{cr}$ , etc) to  $p$ -adic constructions inside  $\mathbb{C}_p$ . In this paper we prove Fontaine’s conjecture by applying methods from [2]. These methods were used earlier to study ramification properties in the characteristic  $p$  case. As a matter of fact, this is the first time when we use them in the mixed characteristic situation.

## 1. CONSTRUCTION OF TORSION CRYSTALLINE REPRESENTATIONS

The ring  $R$  is perfect of characteristic  $p$ , it is provided with the valuation  $v_R$  such that  $v_R(r) := \lim_{n \rightarrow \infty} p^n v_p(o_n)$ , where  $r = (o_n \bmod p)_{n \geq 0}$ . With respect to  $v_R$ ,  $R$  is complete and the field  $R_0 := \text{Frac} R$  is algebraically closed. Note that  $R$  and  $R_0$  are provided with natural  $\Gamma_K$ -action. Denote by  $\sigma$  the Frobenius endomorphism of  $R$  and  $W(R)$  and by  $\mathfrak{m}_R$  the maximal ideal of  $R$ .

1.1. Let  $\mathcal{G} = \text{Spf } W(k)[[X]]$  be the Lubin-Tate 1-dimensional formal group over  $W(k)$  such that  $\text{pid}_{\mathcal{G}}(X) = pX + X^p$ . Then  $\text{End}_{W(k)} \mathcal{G} = \mathbb{Z}_p$  and for any  $l \in \mathbb{Z}_p$ ,  $(\text{id}_{\mathcal{G}})(X) \equiv lX \bmod X^p$ .

Fix  $N \in \mathbb{N}$ .

For  $i \geq 0$ , choose  $o_i \in O_{\mathbb{C}_p}$  such that  $o_0 = 0$ ,  $o_1 \neq 0$  and  $\text{pid}_{\mathcal{G}}(o_{i+1}) = o_i$ . Set  $\tilde{u} = (o_{N+i} \bmod p)_{i \geq 0} \in R$ . Then  $\mathcal{K} := k((\tilde{u}))$  is a complete discrete valuation closed subfield in  $R_0$ . If  $\mathcal{K}_{sep}$  is the separable closure of  $\mathcal{K}$  in  $R_0$  then  $\mathcal{K}_{sep}$  is separably closed and its completion coincides with  $R_0$ . The theory of the field-of-norms functor [7] identifies  $\Gamma_{\mathcal{K}}$  with a closed subgroup in  $\Gamma_K$ . The quotient  $\Gamma_K/\Gamma_{\mathcal{K}}$  acts strictly on  $\mathcal{K}$ . More precisely, there is a group epimorphism  $\kappa : \Gamma_K \rightarrow \text{Aut}_{W(k)} \mathcal{G} \simeq \mathbb{Z}_p^*$  such that if  $g \in \Gamma_K$  then  $\kappa(g) \in \mathbb{Z}_p[[X]]$  and  $\kappa(g)(X) \equiv \chi(g)X \bmod X^p$  with  $\chi(g) \in \mathbb{Z}_p^*$ . (Actually,  $g \mapsto \chi(g)$  is the cyclotomic character.) With this notation we have  $g(\tilde{u}) = \kappa(g)(\tilde{u})$ .

Use the  $p$ -basis  $\{\tilde{u}\}$  for separable extensions  $\mathcal{E}$  of  $\mathcal{K}$  in  $\mathcal{K}_{sep}$  to construct the system of lifts  $O_N(\mathcal{E})$  of  $\mathcal{E}$  modulo  $p^N$ . Recall that  $O_N(\mathcal{E}) = W_N(\sigma^{N-1} \mathcal{E})[u_N] \subset W_N(\mathcal{E})$  and  $O_N(\mathcal{K}) = W_N(k)((u_N))$ , where  $u_N$  is the Teichmüller representative of  $\tilde{u}$  in  $W_N(\mathcal{K})$ . This construction essentially depends on a choice of  $p$ -basis in  $\mathcal{K}$ . If, say,  $\{u'\}$  is another  $p$ -basis for  $\mathcal{K}$  and  $O'_N(\mathcal{E})$  are the appropriate lifts then  $O_N(\mathcal{E})$  and  $O'_N(\mathcal{E})$  are not very much different one from another: they can be related by the natural embeddings  $\sigma^{N-1} O_N(\mathcal{E}) \subset W(\sigma^{N-1} \mathcal{E}) \subset O'_N(\mathcal{E})$ . The lifts  $O_N(\mathcal{E})$  are provided with the endomorphism  $\sigma$  such that  $\sigma u_N = u_N^p$ , and  $O_N(\mathcal{K}_{sep})$  is provided with continuous  $\Gamma_{\mathcal{K}}$ -action.

If  $\tau$  is a continuous automorphism of  $\mathcal{E}$  then generally  $\tau$  can't be lifted to an automorphism of  $O_N(\mathcal{E})$  (but it can always be lifted to  $W_N(\mathcal{E})$ ). In many cases it is sufficient to use “the lift”  $\hat{\tau} : \sigma^{N-1}O_N(\mathcal{E}) \rightarrow O_N(\mathcal{E})$  induced by  $W_N(\tau) : W_N(\mathcal{E}) \rightarrow W_N(\mathcal{E})$ . In other words,  $\hat{\tau}$  is defined only on a part of  $O_N(\mathcal{E})$ , but  $\hat{\tau} \bmod p = \sigma^{N-1} \circ \tau : \sigma^{N-1}\mathcal{E} \rightarrow \sigma^{N-1}\mathcal{E}$  and, therefore,  $\tau$  can be uniquely recovered from the “lift”  $\hat{\tau}$ .

On the other hand, any continuous automorphism  $\tau$  of  $\mathcal{K} = k((\tilde{u}))$  can be lifted to an automorphism  $\tau^{(N)}$  of  $O_N(\mathcal{K}) = W_N(k)((u_N))$  (use that  $u_N \bmod p = \tilde{u}$ ). Taking into account the existence of a lift  $\tau_{sep}$  of  $\tau$  to  $\mathcal{K}_{sep}$  we obtain a lift  $\tau_{sep}^{(N)}$  of  $\tau$  to  $O_N(\mathcal{K}_{sep}) = W_N(\sigma^{N-1}\mathcal{K}_{sep})[u_N]$ .

Set  $O_N^0 := O_N(\mathcal{K}_{sep}) \cap W_N(O_{sep})$  and  $O_N^+ := O_N(\mathcal{K}_{sep}) \cap W_N(\mathfrak{m}_{sep})$ , where  $\mathfrak{m}_{sep}$  is the maximal ideal of the valuation ring  $O_{sep}$  of  $\mathcal{K}_{sep}$ . Then  $\sigma(O_N^0) \subset O_N^0$ ,  $\sigma(O_N^+) \subset O_N^+$  and  $\bigcap_{n \geq 0} \sigma^n(O_N^+) = 0$ . Note that  $O_N^0(\mathcal{K}) := O_N^0 \cap O_N(\mathcal{K}) = W_N(k)[[u_N]]$ ,  $O_N^+(\mathcal{K}) := O_N^+ \cap O_N(\mathcal{K}) = u_N W_N(k)[[u_N]]$  and  $O_N(\mathcal{K}) = O_N^0(\mathcal{K})[u_N^{-1}] = W_N(k)((u_N))$ .

For  $0 \leq m \leq N$ , introduce

$$u_m = (p^{N-m} \text{id}_{\mathcal{G}})(u_N) \in O_N^0(\mathcal{K})$$

Then  $u_0 = \sigma u_1 = pu_1 + u_1^p$ ,  $t = u_0/u_1 = p + u_1^{p-1} \in O_N^0(\mathcal{K})$  and  $u_0^{p-1} = t^p - pt^{p-1}$ . As a matter of fact,  $u_0, u_1, t$  depend only on  $\tilde{u}$ . Indeed, if  $u' \in W_N(R)$  and  $u' \bmod pW_N(R) = \tilde{u}$  then in  $O_N(\mathcal{K})$  we have  $u_1 = (p^{N-1} \text{id}_{\mathcal{G}})(u')$ .

**Lemma 1.1.** *Suppose  $g \in \Gamma_K$ . Then*

- a)  $g(u_0) \equiv \chi(g)u_0 \bmod u_0^p O_N^0(\mathcal{K})$ ;
- b)  $\sigma(g(t)/t) \equiv 1 \bmod u_0^{p-1} O_N^0(\mathcal{K})$ .

*Proof.*  $g(u_1) = (p^{N-1} \text{id}_{\mathcal{G}})(g(u_N)) = \kappa(g)(u_1) \equiv \chi(g)u_1 \bmod u_1^p O_N^0(\mathcal{K})$  implies a) because  $\sigma(u_1) = u_0$ . Then  $g(t)/t \equiv 1 \bmod u_1^{p-1} O_N^0(\mathcal{K})$  and applying  $\sigma$  we obtain b).  $\square$

1.2. Let  $\mathcal{MF}$  be the category of  $W(k)$ -modules  $M$  provided with decreasing filtration by  $W(k)$ -submodules  $M = M^0 \supset \dots \supset M^{p-1} \supset M^p = 0$  and  $\sigma$ -linear morphisms  $\varphi_i : M^i \rightarrow M$  such that for all  $i$ ,  $\varphi_i|_{M^{i+1}} = p\varphi_{i+1}$ .

For  $0 \leq a \leq p-2$ , introduce the filtered module  $\mathcal{S}_a$  such that

- $\mathcal{S}_a = O_N^0/u_0^a O_N^+$ ;
- for  $0 \leq i \leq a$ ,  $\text{Fil}^i \mathcal{S}_a = t^i \mathcal{S}_a$ ;
- $\varphi_i : \text{Fil}^i \mathcal{S}_a \rightarrow \mathcal{S}_a$  is  $\sigma$ -linear morphism such that  $\varphi_i(t^i) = 1$ .

Clearly,  $\mathcal{S}_a \in \mathcal{MF}$  (use that  $\sigma t \equiv p \bmod u_0^{p-1}$ ). In addition, Lemma 1.1 implies also that the action of  $\Gamma_K$  preserves the structure of an object of the category  $\mathcal{MF}$  on  $\mathcal{S}_a$ .

For  $0 \leq a < p$ , define the category of filtered Fontaine-Laffaille modules  $\text{MF}_N(a)$  as the full subcategory in  $\mathcal{MF}$  consisting of modules

$M$  of finite length over  $W_N(k)$  such that  $M^{a+1} = 0$  and  $\sum \text{Im} \varphi_i = M$ . We can assume that  $M$  is given together with a functorial splitting of its filtration, i.e. there are submodules  $N_i$  in  $M$  such that for all  $i$ ,  $M^i = N_i \oplus M^{i+1}$ .

Let  $M \in \text{MF}_N(a)$  and  $\tilde{U}_a(M) = \text{Hom}_{\mathcal{MF}}(M, \mathcal{S}_a)$ . Then the correspondence  $M \mapsto \tilde{U}_a(M)$  determines the functor  $\tilde{U}_a$  from  $\text{MF}_N(a)$  to the category of  $\Gamma_K$ -modules.

**Proposition 1.2.** *If  $0 \leq a \leq p-2$  and  $H \in \text{MF}_N^{cr}(a)$  then there is  $M \in \text{MF}_N(a)$  such that  $\tilde{U}_a(M) = H$ .*

*Proof.* Recall briefly the main ingredients of the Fontaine-Laffaille theory [5]. The  $p^N$ -torsion crystalline ring  $A_{cr,N} := A_{cr}/p^N$  appears as the divided power envelope of  $W_N(R)$  with respect to  $\text{Ker} \gamma$ . We need the following construction of a generator of  $\text{Ker} \gamma$ . (Note that we have a natural inclusion of  $W(k)$ -modules  $O_N^0 \subset W_N(R)$ .)

**Lemma 1.3.**  $\text{Ker} \gamma = tW_N(R)$ .

*Proof.* We have  $\gamma(u_N) \equiv o_N \pmod{pO_{\mathbb{C}_p}}$ , therefore,  $\gamma(u_0) \equiv 0 \pmod{p^N O_{\mathbb{C}_p}}$  and  $t \in \text{Ker} \gamma$ . On the other hand,  $t \equiv u_1^{p-1} \equiv [r] \pmod{pW(R)}$ , where  $r \in R$  is such that  $r^{(0)} \equiv o_1^{p-1} \equiv -p \pmod{p^{p/(p-1)} O_{\mathbb{C}_p}}$ . Therefore,  $v_p(r^{(0)}) = 1$  and  $t$  generates  $\text{Ker} \gamma$ , cf. [5].  $\square$

By above Lemma,  $A_{cr,N} = W_N(R)[\{\gamma_i(t) \mid i \geq 1\}]$ , where  $\gamma_i(t)$  are the  $i$ -th divided powers of  $t$ . Then the identity  $\gamma_p(t) = t^{p-1} + u_0^{p-1}/p$  implies that  $A_{cr,N} = W_N(R)[\{\gamma_i(u_0^{p-1}/p) \mid i \geq 1\}]$ .

Recall that  $A_{cr,N} \in \mathcal{MF}$  with:

— the filtration  $\text{Fil}^i A_{cr,N}$ ,  $0 \leq i < p$ , generated as ideal by  $t^i$  and all  $\gamma_j(u_0^{p-1}/p)$ ,  $j \geq 1$ ;

— the  $\sigma$ -linear morphisms  $\varphi_i : \text{Fil}^i A_{cr,N} \rightarrow A_{cr,N}$  (which come from  $\sigma/p^i$  on  $A_{cr}$ ) such that  $\varphi_i(t^i) = (1 + u_0^{p-1}/p)^i$  and  $\varphi_i(u_0^{p-1}/p) = p^{p-1-i}(u_0^{p-1}/p)(1 + u_0^{p-1}/p)^{p-1}$ .

Then the Fontaine-Laffaille functor  $U_a$  attaches to  $M \in \text{MF}_N(a)$  the  $\Gamma_K$ -module  $\text{Hom}_{\mathcal{MF}}(M, A_{cr,N})$ . This functor is fully-faithful (we assume that  $a \leq p-2$ ) and, therefore, there is  $M \in \text{MF}_N(a)$  such that  $U_a(M) = H$ .

Consider the  $W(k)$ -module  $\mathcal{W}_N^a = W_N(R)/u_0^a W_N(\mathfrak{m}_R)$  with the filtration induced by the filtration  $W_N^i(R) = t^i W_N(R)$  and  $\sigma$ -linear morphisms  $\varphi_i$  such that  $\varphi_i(t^i) = 1$ . Prove that we have an identification of  $\Gamma_K$ -modules  $H = \text{Hom}_{\mathcal{MF}}(M, \mathcal{W}_N^a)$ .

Indeed, let  $T_a$  be the maximal element in the family of all ideals  $I$  of  $A_{cr,N}$  such that  $\varphi_a$  induces a nilpotent endomorphism of  $I$ . Then for any  $M \in \text{MF}_N(a)$ ,  $U_a(M) = \text{Hom}_{\mathcal{MF}}(M, A_{cr,N}/T_a)$ . By straightforward calculations we can see that  $T_a$  is generated by the elements of

$u_0^a W_N(\mathfrak{m}_R)$  and all  $\gamma_j(u_0^{p-1}/p)$ ,  $j \geq 1$ . It remains to note that we have a natural identification  $A_{cr,N}/T_a = \mathcal{W}_N^a$  in the category  $\mathcal{MF}$ .

Consider the natural embedding  $O_N^0 \rightarrow W_N(R)$  and the induced natural map  $\iota_a : \mathcal{S}_a \rightarrow \mathcal{W}_N^a$  in  $\mathcal{MF}$ . Prove that  $\iota_{a*} : \tilde{U}_a(M) \rightarrow H$  is isomorphism of  $\Gamma_K$ -modules.

Choose  $W(k)$ -submodules  $N_i$  in  $M^i$  such that  $M^i = N_i \oplus M^{i+1}$  and choose vectors  $\bar{n}_i$  whose coordinates give a minimal system of generators of  $N_i$ . Then the structure of  $M$  can be given by the matrix relation  $(\varphi_a(\bar{n}_a), \dots, \varphi_0(\bar{n}_0)) = (\bar{n}_a, \dots, \bar{n}_0)C$ , where  $C$  is an invertible matrix with coefficients in  $W(k)$ . The elements of  $H$  are identified with the residues  $(\bar{u}_a, \dots, \bar{u}_0) \bmod u_0^a W_N(\mathfrak{m}_{sep})$  where the vectors  $(\bar{u}_a, \dots, \bar{u}_0)$  have coefficients in  $W_N(\mathcal{K}_{sep})$  and satisfy the following system of equations (use that  $\varphi_a$  is topologically nilpotent on  $u_N^a W_N(\mathfrak{m}_{sep})$ )

$$\left( \frac{\sigma \bar{u}_a}{\sigma t^a}, \dots, \frac{\sigma \bar{u}_i}{\sigma t^i}, \dots, \sigma(\bar{u}_0) \right) = (\bar{u}_a, \dots, \bar{u}_0)C$$

In particular, if  $\bar{u} = (\bar{u}_a, \dots, \bar{u}_0)$  then there is an invertible matrix  $D$  with coefficients in  $O_N(\mathcal{K})$  such that

$$(1.1) \quad \sigma(\bar{u})D = \bar{u}.$$

We know that all coordinates of  $\sigma^{N-1}\bar{u}$  belong to  $\sigma^{N-1}W_N(\mathcal{K}_{sep}) \subset O_N(\mathcal{K}_{sep})$ . Then (1.1) implies step-by-step that the vectors  $\sigma^{N-2}\bar{u}, \dots, \bar{u}$  have coordinates in  $O_N(\mathcal{K}_{sep})$ . It remains to note that  $O_N^0 = O_N(\mathcal{K}_{sep}) \cap W_N(O_{sep})$  and  $O_N^+ = O_N(\mathcal{K}_{sep}) \cap W_N(\mathfrak{m}_{sep})$ . The proposition is proved.  $\square$

## 2. REFORMULATION OF THE FONTAINE CONJECTURE

**2.1. Review of ramification theory.** Let  $\mathcal{I}_{\mathcal{K}}$  be the group of all continuous automorphisms of  $\mathcal{K}_{sep}$  which keep invariant the residue field of  $\mathcal{K}_{sep}$  and preserve the extension of the normalised valuation  $v_{\mathcal{K}}$  of  $\mathcal{K}$  to  $\mathcal{K}_{sep}$ . This group has a decreasing filtration by its ramification subgroups  $\mathcal{I}_{\mathcal{K}}^{(v)}$  in upper numbering  $v \geq 0$ . Recall basic ingredients of the definition of this filtration following the papers [3, 7, 8].

For any field extension  $\mathcal{E}$  of  $\mathcal{K}$  in  $\mathcal{K}_{sep}$ , set  $\mathcal{E}_{sep} = \mathcal{K}_{sep}$ , in particular,  $\mathcal{I}_{\mathcal{E}} = \mathcal{I}_{\mathcal{K}}$ . All elements of  $\mathcal{I}_{\mathcal{K}}$  preserve the extension  $v_{\mathcal{E}}$  of the normalised valuation on  $\mathcal{E}$  to  $\mathcal{K}_{sep}$ .

For  $x \geq 0$ , set  $\mathcal{I}_{\mathcal{E},x} = \{\iota \in \mathcal{I}_{\mathcal{E}} \mid v_{\mathcal{E}}(\iota(a) - a) \geq 1+x \ \forall a \in \mathfrak{m}_{\mathcal{E}}\}$ , where  $\mathfrak{m}_{\mathcal{E}}$  is the maximal ideal in  $O_{\mathcal{E}}$ .

Denote by  $\mathcal{I}_{\mathcal{E}/\mathcal{K}}$  the set of all continuous embeddings of  $\mathcal{E}$  into  $\mathcal{K}_{sep}$  which induce the identity map on  $\mathcal{K}$  and the residue field  $k_{\mathcal{E}}$  of  $\mathcal{E}$ . For  $x \geq 0$ , set  $\mathcal{I}_{\mathcal{E}/\mathcal{K},x} = \mathcal{I}_{\mathcal{E},x} \cap \mathcal{I}_{\mathcal{E}/\mathcal{K}}$ .

If  $\iota_1, \iota_2 \in \mathcal{I}_{\mathcal{E}/\mathcal{K}}$  and  $x \geq 0$  then  $\iota_1$  and  $\iota_2$  are  $x$ -equivalent iff for any  $a \in \mathfrak{m}_{\mathcal{E}}$ ,  $v_{\mathcal{E}}(\iota_1(a) - \iota_2(a)) \geq 1+x$ . Denote by  $(\mathcal{I}_{\mathcal{E}/\mathcal{K}} : \mathcal{I}_{\mathcal{E}/\mathcal{K},x})$  the number of  $x$ -equivalent classes in  $\mathcal{I}_{\mathcal{E}/\mathcal{K}}$ . Then the Herbrand function  $\varphi_{\mathcal{E}/\mathcal{K}}$  can

be defined for all  $x \geq 0$ , as

$$\varphi_{\mathcal{E}/\mathcal{K}}(x) = \int_0^x (\mathcal{I}_{\mathcal{E}/\mathcal{K}} : \mathcal{I}_{\mathcal{E}/\mathcal{K},x})^{-1} dx.$$

This function has the following properties:

- $\varphi_{\mathcal{E}/\mathcal{K}}$  is a piece-wise linear function with finitely many edges;
- if  $\mathcal{K} \subset \mathcal{E} \subset \mathcal{H}$  is a tower of finite field extensions in  $\mathcal{K}_{sep}$  then for any  $x \geq 0$ ,  $\varphi_{\mathcal{H}/\mathcal{K}}(x) = \varphi_{\mathcal{E}/\mathcal{K}}(\varphi_{\mathcal{H}/\mathcal{E}}(x))$ .

The ramification filtration  $\{\mathcal{I}_{\mathcal{K}}^{(v)}\}_{v \geq 0}$  appears now as a decreasing sequence of the subgroups  $\mathcal{I}_{\mathcal{K}}^{(v)}$  of  $\mathcal{I}_{\mathcal{K}}$ , where  $\mathcal{I}_{\mathcal{K}}^{(v)}$  consists of  $\iota \in \mathcal{I}_{\mathcal{K}}$  such that for any finite extension  $\mathcal{E}$  of  $\mathcal{K}$ ,  $\iota \in \mathcal{I}_{\mathcal{E}/\mathcal{K}}$  with  $\varphi_{\mathcal{E}/\mathcal{K}}(v_{\mathcal{E}}) = v$ .

If we replace the lower indices  $\mathcal{K}$  to  $\mathcal{E}$ , the ramification filtration  $\{\mathcal{I}_{\mathcal{K}}^{(v)}\}_{v \geq 0}$  is not changed as a whole, just only individual subgroups change their upper indices, that is  $\mathcal{I}_{\mathcal{K}}^{(v)} = \mathcal{I}_{\mathcal{E}}^{(v_{\mathcal{E}})}$ .

Note that the inertia subgroup  $\Gamma_{\mathcal{E}}^0$  of  $\Gamma_{\mathcal{E}} = \text{Gal}(\mathcal{K}_{sep}/\mathcal{E})$  is a subgroup in  $\mathcal{I}_{\mathcal{E}}$  and for any  $v \geq 0$ , the appropriate subgroup  $\Gamma_{\mathcal{E}}^{(v)} = \Gamma_{\mathcal{E}} \cap \mathcal{I}_{\mathcal{E}}^{(v)}$  is just the ramification subgroup of  $\Gamma_{\mathcal{E}}$  with the upper number  $v$  from [6].

**2.2. Statement of the main theorem.** The main idea of our approach to the  $\Gamma_K$ -modules  $\tilde{U}_a(M)$  is related to the following fact. The filtered module  $\mathcal{S}_a$  depends only on the field  $\mathcal{K}$  and its uniformizer  $\tilde{u}$ . Therefore,  $\mathcal{S}_a$  can be identified with its analogue  $\mathcal{S}'_a$  constructed for any ramified extension  $\mathcal{K}'$  of  $\mathcal{K}$  together with its uniformizer  $\tilde{u}'$ . The whole group  $\mathcal{I}_{\mathcal{K}}$  does not preserve the structure of  $\mathcal{S}_a$  but the ramification subgroups  $\mathcal{I}_{\mathcal{K}}^{(v)}$ , where  $a > a_N^* := (a+1)p^{N-1} - 1$  do preserve this structure because of the following proposition.

**Proposition 2.1.** *If  $v > a_N^*$  and  $M \in \text{MF}_N(a)$  then a natural action of  $\mathcal{I}_{\mathcal{K}}$  on  $W_N(\mathcal{K}_{sep})$  induces the  $\mathcal{I}_{\mathcal{K}}^{(v)}$ -module structure on  $\tilde{U}_a(M)$ .*

*Proof.* All we need is just the following lemma. □

**Lemma 2.2.** *If  $\tau \in \mathcal{I}_{\mathcal{K}}^{(v)}$  with  $v > a_N^*$  then*

- $\tau(u_0)/u_0 \in O_N^*(\mathcal{K}_{sep})$ ;
- for  $0 \leq i \leq a$ ,  $\varphi_i(\tau t^i) = 1$ .

*Proof of Lemma.* For  $\tau \in \mathcal{I}_{\mathcal{K}}^{(v)}$ , we have  $\tau(u_N) = u_N + \eta_N + pw$ , where  $\eta_N \in u_1^{a+1}O_N^+$  and  $w \in W_N(\mathcal{K}_{sep})$ . For  $1 \leq i \leq N$ , this implies

$$\tau(u_i) = u_i + \eta_i + p^{N-i+1}w_i,$$

where  $\eta_i \in u_1^{a+1}O_N^+$  and  $w_i \in W_N(\mathcal{K}_{sep})$ . Therefore,

$$\tau(u_1) \equiv u_1 \pmod{u_1^{a+1}O_N^+}.$$

This implies part a) because  $\tau(u_0) \equiv u_0 \pmod{u_0^{a+1}O_N^+}$  and part b) because  $\sigma(\tau t)/\sigma(t) \equiv 1 \pmod{u_0^a O_N^+}$ . □

With the relation to the original problem of estimating the upper ramification numbers of the  $\Gamma_K$ -module  $H$  notice now that  $\mathcal{K} = k((\tilde{u}))$  coincides with  $\sigma^{-N}\mathcal{K}_0$ , where  $\mathcal{K}_0$  is the field-of-norms of the  $p$ -cyclotomic extension  $\tilde{K}$  of  $K$ . Then for any  $v \geq 0$ ,  $\Gamma_K^{(v)} = \Gamma_K \cap \mathcal{I}_{\mathcal{K}}^{(v^*)}$ , where  $\varphi_{\tilde{K}/K}(v^*) = v$ . In particular,  $v > N - 1 + a/(p - 1)$  if and only if  $v^* > a_N^*$ .

So, the proof of Fontaine's conjecture is reduced to the proof of the following theorem stated exclusively in terms of the field  $\mathcal{K}$  of characteristic  $p$ .

**Theorem 2.3.** *For any  $v > a_N^*$ , the group  $\mathcal{I}_{\mathcal{K}}^{(v)}$  acts trivially on  $\tilde{U}_a(M)$ .*

### 3. PROOF OF THEOREM 2.3

**3.1. Auxiliary field  $\mathcal{K}'$ .** Let  $N^* \in \mathbb{N}$  and  $r^* \in \mathbb{Q}$  be such that for  $q := p^{N^*}$ ,  $r^*(q - 1) := b^* \in \mathbb{N}$  and  $v_p(b^*) = 0$ .

Consider the field  $\mathcal{K}' = \mathcal{K}(N^*, r^*)$  from [2]. Remind that

- $[\mathcal{K}' : \mathcal{K}] = q$ ;
- $\mathcal{K}' = k((\tilde{u}'))$ , where  $\tilde{u} = \tilde{u}'^q E(\tilde{u}'^{b^*})^{-1}$  (here  $E$  is the Artin-Hasse exponential);
- the Herbrand function  $\varphi_{\mathcal{K}'/\mathcal{K}}$  has only one edge point  $(r^*, r^*)$ .

For  $\mathcal{K}'$  and its above uniformiser  $\tilde{u}'$  proceed as earlier to construct the lifts  $O'_N(\mathcal{K}')$  and  $O'_N(\mathcal{K}_{sep})$  obtained with respect to the  $p$ -basis  $\tilde{u}'$ . Introduce similarly the modules  $O_N'^0, O_N'^+$ , the elements  $u'_0, t' \in O_N(\mathcal{K}')$  and the filtered module  $\mathcal{S}'_a$ .

**3.2.** Compare the old and the new lifts using their canonical embeddings into  $W_N(\mathcal{K}_{sep})$ . Note that  $u_N$  is not generally an element of  $O'_N(\mathcal{K}')$  because the Teichmüller representative  $u_N = [\tilde{u}]$  can't be written as a power series in  $u'_N = [\tilde{u}']$  if  $N > 1$ . However, we can easily see that for  $1 \leq i < N$ ,  $u_{N-i} \in O'_N(\mathcal{K}') \bmod p^{i+1}W_N(\mathcal{K}')$ . In particular,  $u_1, u_0, t \in O'_N(\mathcal{K}')$ .

**Proposition 3.1.** *If  $\xi \in \tilde{U}_a(M)$  then for any  $m \in M$ ,  $\xi(m) \in O'_N(\mathcal{K}_{sep})$ .*

*Proof.* Proceed as we proceeded at the end of Section 1. Then the vectors  $(\xi(\bar{n}_a), \dots, \xi(\bar{n}_0))$  appear in the form  $\bar{\xi} \bmod u_0^a O_N^+$ , where  $\bar{\xi}$  is a vector with coefficients in  $O_N(\mathcal{K}_{sep})$  such that

$$(3.1) \quad \sigma(\bar{\xi})D = \bar{\xi},$$

and the matrix  $D$  has coefficients in  $O'_N(\mathcal{K}')$  (use that  $t \in O'_N(\mathcal{K}')$ ). We know that all coordinates of  $\sigma^{N-1}\bar{\xi}$  belong to  $\sigma^{N-1}O_N(\mathcal{K}_{sep}) \subset O'_N(\mathcal{K}_{sep})$ . Then (3.1) implies step-by-step that the vectors  $\sigma^{N-2}\bar{\xi}, \dots, \bar{\xi}$  have coordinates in  $O'_N(\mathcal{K}_{sep})$ .  $\square$

3.3. Now suppose  $v^* \in \mathbb{Q}$ ,  $\mathcal{I}_{\mathcal{K}}^{(v)}$  acts trivially on  $\tilde{U}_a(M)$  for all  $v > v^*$  and  $v^*$  is the minimal with this property. The existence of  $v^*$  follows from the left-continuity of the ramification filtration with respect to the upper numbering.

If  $v^* \leq a_N^*$  then our theorem is proved.

Suppose that  $v^* > a_N^*$ . Choose the parameters  $r^*$  and  $N^*$  from Subsection 3.1 such that  $a_N^* q / (q - 1) < r^* < v^*$ .

For any  $\alpha \in O'_N(\mathcal{K}_{sep})$ , set  $\alpha^{(q)} = \sigma^{N^*} \alpha$ .

**Lemma 3.2.**  $u_1 / u_1'^{(q)} \equiv 1 \pmod{u_1'^{(q)a} O_N'^+(\mathcal{K})}$ .

*Proof.* Consider  $b^* = r^*(q - 1) \in \mathbb{N}$  from Subsection 3.1. Then  $b^* + q > q(a_N^* + 1) = q(a + 1)p^{N-1}$  and

$$u_N \equiv u_N'^{(q)} \pmod{\left(u_1'^{(q)a+1} O_N^+(\mathcal{K}') + p O_N(\mathcal{K}')\right)}$$

This implies  $u_1 \equiv u_1'^{(q)} \pmod{u_1'^{(q)a+1} O_N'^+(\mathcal{K})}$  and the lemma is proved.  $\square$

**Corollary 3.3.** a)  $u_0 / u_0'^{(q)}$  is invertible in  $O_N'^0(\mathcal{K})$ ;

b)  $\sigma(t/t_0'^{(q)}) \equiv 1 \pmod{u_0'^{(q)a} O_N'^+(\mathcal{K})}$ .

3.4.  $\mathcal{I}_{\mathcal{K}'}^{(v^*)}$ -**action.** Introduce the filtered module  $\mathcal{S}_a'^{(q)}$  as follows.

- $\mathcal{S}_a'^{(q)} = O_N'^0 / u_0'^{(q)a} O_N'^+$ ;
- for  $0 \leq i \leq a$ ,  $\text{Fil}^i \mathcal{S}_a'^{(q)} = t'^{(q)i} \mathcal{S}_a'^{(q)}$ ;
- $\varphi_i'^{(q)} : \text{Fil}^i \mathcal{S}_a'^{(q)} \rightarrow \mathcal{S}_a'^{(q)}$  is  $\sigma$ -linear such that  $\varphi_i'^{(q)}(t'^{(q)i}) = 1$ .

Suppose  $M' \in \text{MF}_N(a)$  is given similarly to  $M$  by the relation

$$(\varphi_a(\bar{n}_a), \dots, \varphi_0(\bar{n}_0)) = (\bar{n}_a, \dots, \bar{n}_0) \sigma^{-N^*} C$$

Then we can use  $\sigma^{N^*}$  to identify the modules  $\tilde{U}_a'(M') := \text{Hom}_{\mathcal{MF}}(M', \mathcal{S}_a')$  and  $\tilde{U}_a'^{(q)}(M) := \text{Hom}_{\mathcal{MF}}(M, \mathcal{S}_a'^{(q)})$ . This identification is compatible with the action of the subgroups  $\mathcal{I}_{\mathcal{K}'}^{(v)}$ , where  $v > a_N^*$ .

Note that the fields  $\mathcal{K}$  and  $\mathcal{K}'$  are isomorphic (as any two fields of formal power series with the same residue field). Choose an isomorphism  $\kappa : \mathcal{K} \rightarrow \mathcal{K}'$  such that  $\kappa(\tilde{u}) = \tilde{u}'$  and  $\kappa|_k = \sigma^{-N^*}$ . We can extend  $\kappa$  to an isomorphism of separable closures of  $\mathcal{K}$  and  $\mathcal{K}'$ . This allows us to identify the groups  $\mathcal{I}_{\mathcal{K}}$  and  $\mathcal{I}_{\mathcal{K}'}$  and this identification is compatible with the appropriate ramification filtrations. Even more, we obtain an identification of  $\tilde{U}_a(M)$  with  $\tilde{U}_a'(M')$  and this identification respects the action of  $\mathcal{I}_{\mathcal{K}}^{(v)}$  on  $\tilde{U}_a(M)$  and the action of  $\mathcal{I}_{\mathcal{K}'}^{(v)}$  on  $\tilde{U}_a'(M')$  for any  $v > a_N^*$ . Therefore,  $v^*$  is the maximal number such that  $\mathcal{I}_{\mathcal{K}'}^{(v^*)}$  acts non-trivially on  $\tilde{U}_a'(M')$  and

- $v^*$  is the maximal such that  $\mathcal{I}_{\mathcal{K}'}^{(v^*)}$  acts non-trivially on  $\tilde{U}_a'^{(q)}(M)$ .



3.5.  $\mathcal{I}_{\mathcal{K}}^{(v^*)}$ -**action.** Introduce the filtered module  $\mathcal{S}_a^*$  as follows:

- $\mathcal{S}_a^* = O_N^0 \cap O'_N(\mathcal{K}_{sep})/u_0^a O_N^+ \cap O'_N(\mathcal{K}_{sep})$ ;
- $\text{Fil}^i \mathcal{S}_a^* = t^i \mathcal{S}_a \cap \mathcal{S}_a^*$ ;
- $\varphi_i^* = \varphi_i|_{\text{Fil}^i \mathcal{S}_a^*} : \text{Fil}^i \mathcal{S}_a^* \longrightarrow \mathcal{S}_a^*$ .

The results from Subsection 3.2 allow us to identify  $\tilde{U}_a(M)$  with  $U_a^*(M) = \text{Hom}_{\mathcal{MF}}(M, \mathcal{S}_a^*)$ . By the results from Subsection 3.3, there is a natural embedding of filtered modules  $\mathcal{S}_a^* \longrightarrow \mathcal{S}_a'^{(q)}$  and, therefore, we can identify  $\tilde{U}_a(M)$  with  $\tilde{U}_a'^{(q)}(M')$ . This identification is compatible with the action of ramification subgroups  $\mathcal{I}_{\mathcal{K}}^{(v)}$  for all  $v > a_N^*$ . So,

- $v^*$  is the maximal such that  $\mathcal{I}_{\mathcal{K}}^{(v^*)}$  acts non-trivially on  $\tilde{U}_a'^{(q)}(M)$ .

3.6. **The end of proof of Theorem.** It remains to notice that  $\mathcal{I}_{\mathcal{K}'}^{(v^*)} = \mathcal{I}_{\mathcal{K}}^{(v_0^*)}$ , where  $v_0^* = \varphi_{\mathcal{K}'/\mathcal{K}}(v^*) < v^*$ .

The contradiction.

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